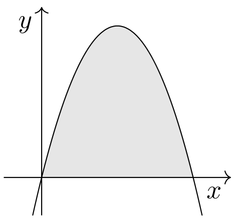
2601. The region is question is



For the t limits, we solve $4 - t^2 = 0$, so $t = \pm 2$. These increase with x, so, using the parametric area formula, the integral is

$$\begin{aligned} &\int_{t_1}^{t_2} y \frac{dx}{dt}\, dt \\ &= \int_{-2}^{2} \left(4 - t^2\right) \cdot 1\, dt \\ &= \left[4t - \tfrac{1}{3}t^3\right]_{-2}^{2} \\ &= \left(8 - \tfrac{8}{3}\right) - \left(-8 + \tfrac{8}{3}\right) \\ &= \tfrac{32}{3}. \end{aligned}$$

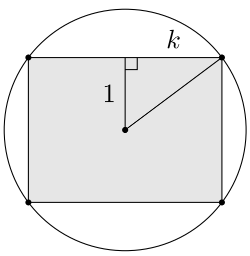
— Nota Bene —

In the above solution, we noted that t *increases* with x because a parametric area integral has two different ways of generating negative value, unlike a standard integral which has one.

① In both parametric and standard integrals (with respect to x), if y is negative, then this contributes negatively to the integral. This is the usual fact that an area below the x axis registers as a negative signed area.

② In a parametric integral, if $\frac{dx}{dt}$ is negative, i.e. if we are moving in the negative x direction (leftwards) with t, then this also contributes negatively to the integral, even if the relevant area is above the x axis.

2602. Since the sides are in the ratio $1 : k$, so are the half-sides:



The square of the radius is then $1 + k^2$. So, the area of the circle is $\pi\left(1 + k^2\right)$. And the area of the rectangle is $4k$. Hence, the ratio of areas is $4k : \pi\left(1 + k^2\right)$, as required.

2603. We require a non-zero denominator:

$$\begin{aligned} & x^2 - |x| = 0 \\ \Longleftrightarrow\ & x^2 = |x| \\ \Longleftrightarrow\ & x^4 = x^2 \\ \Longleftrightarrow\ & x^2(x+1)(x-1) = 0 \\ \Longleftrightarrow\ & x = -1, 0, 1. \end{aligned}$$

These values must be excluded. This leaves the largest possible domain as $\mathbb{R} \setminus \{-1, 0, 1\}$.

2604. (a) Let μ be the mean of the population of pseudo-random numbers produced by the generator. Then our hypotheses are

$$\begin{aligned} H_0 &: \mu = 0 \\ H_1 &: \mu > 0. \end{aligned}$$

(b) Assuming H_0, the distribution of the means of samples of size 100 is

$$\bar{Z} \sim N(0, 1/100).$$

This is a one-tailed test at the 5% level, so the critical value is $1.645 \times \sqrt{1/100} = 0.1645$. The test statistic is $0.253 > 0.1645$. This is (highly) significant. There is sufficient evidence at the 5% level to reject H_0. It is (highly) likely that there is indeed a bug.

2605. Writing the integrand in partial fractions,

$$\begin{aligned} & \frac{2}{4 - x^2} \equiv \frac{A}{2 + x} + \frac{B}{2 - x} \\ \Longrightarrow\ & 2 \equiv A(2 - x) + B(2 + x). \end{aligned}$$

Equating coefficients, we have $2 = 2A + 2B$ and $0 = -A + B$. So $A = B = \frac{1}{2}$. This gives

$$\begin{aligned} &\tfrac{1}{2} \int_{-1}^{1} \frac{1}{2 + x} + \frac{1}{2 - x}\, dx \\ &= \tfrac{1}{2} \Big[\ln|2 + x| - \ln|2 - x| \Big]_{-1}^{1} \\ &= \tfrac{1}{2}\big((\ln 3 - \ln 1) - (\ln 1 - \ln 3)\big) \\ &= \ln 3, \text{ as required.} \end{aligned}$$

2606. (a) Classified by the number of SPs, cubics are of one of three types:

① $y = x^3 - x$ (2 SPs),
② $y = x^3$ (1 SP),
③ $y = x^3 + x$ (0 SPs).

Only ② has a stationary point of inflection. So, consider $y = \mathrm{f}(x)$ as a transformed $y = x^3$. Firstly, stretch to $y = px^3$, then translate by $2\mathbf{i} + 4\mathbf{j}$. This gives

$$y = p(x - 2)^3 + 4.$$

If this is to pass through the origin, we need $0 = -8p + 4$, so $p = 1/2$.

(b) Using the binomial expansion, the function f is $\mathrm{f}(x) = \frac{1}{2}x^3 - 3x^2 + 6x$.

2607. In both cases, a many-to-one trig function must be restricted to a one-to-one invertible version.

(a) The range of sin is $[-1, 1]$, so this must be the domain of arcsin.

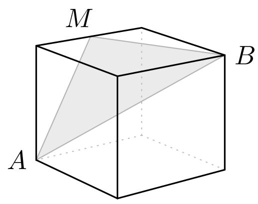
(b) Likewise, the domain of arccos is $[-1, 1]$. The equations $2x + 1 = -1, 1$ give $x = -1, 0$. So, the domain of $\arccos(2x + 1)$ is $[-1, 0]$.

2608. The student has included a second root in the last line, when it should appear in the second line. It appears at the moment the sin function is undone. So, the second line should read $2\theta = 60°, 120°$. This gives $\theta = 30°, 60°$.

2609. The possibility space is a $12 \times n$ rectangle, which contains $12n$ outcomes. If the score on the n-sided die is k, then there are $k - 1$ successful outcomes on the 12-sided die. Hence, the total number of successful outcomes is $1 + 2 + ... + n - 1$. This is an AP with sum $\frac{1}{2}n(n - 1)$. So,

$$\begin{aligned} p &= \frac{\frac{1}{2}n(n-1)}{12n} \\ &\equiv \frac{n-1}{24}. \end{aligned}$$

2610. To begin, let $l = 1$.



By Pythagoras, $\triangle AMB$ is isosceles with lengths

$$\left(\sqrt{2}, \tfrac{\sqrt{5}}{2}, \tfrac{\sqrt{5}}{2}\right).$$

We can split it into two right-angled triangles. These have sides $\sqrt{2}/2$, $\sqrt{5}/2$, and, by Pythagoras, $\sqrt{3}/2$. So, the area of $\triangle AMB$ is

$$A = \tfrac{\sqrt{3}}{2} \times \tfrac{\sqrt{2}}{2} = \tfrac{\sqrt{6}}{4}.$$

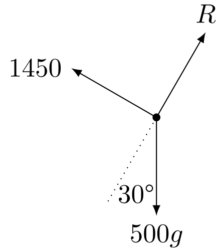
Scaling the cube from side length 1 to l, the area scales by l^2. This gives

$$A_\triangle = \tfrac{\sqrt{6}}{4} l^2, \text{ as required.}$$

2611. This is a quadratic in $\sec x$:

$$\begin{aligned} &\sec^2 x + \sec x - 2 = 0 \\ \Longrightarrow\ &(\sec x + 2)(\sec x - 1) = 0 \\ \Longrightarrow\ &\sec x = -2, 1 \\ \Longrightarrow\ &\cos x = -\tfrac{1}{2}, 1 \\ \therefore\ \ &x = 0, \tfrac{2\pi}{3}, \tfrac{4\pi}{3}. \end{aligned}$$

2612. (a) In the limiting case, the string is taut but with zero tension. The forces on the car are



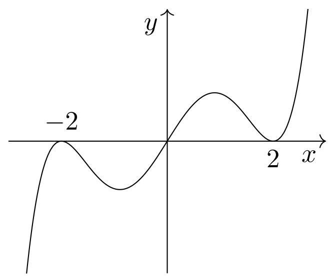
Parallel to the slope, the resultant force on the car is $500g \sin 30° - 1450 = 1000 > 0$ N. So, the car cannot travel at constant speed down the slope, but must accelerate.

(b) Again in the same limiting case, the vehicles have the same acceleration and $T = 0$. So,

$$a_{\text{car}} = a_{\text{jeep}} = \tfrac{1000}{500} = 2 \text{ ms}^{-2}.$$

For minimum final speed, the car must start at rest at the top of the slope, so $v^2 = 2 \cdot 2 \cdot 10$, giving $v = \sqrt{40} = 6.32 \text{ ms}^{-1}$ (3sf).

2613. Factorising, we have $y = x(x-2)^2(x+2)^2$. This is a positive quintic with a single root at $x = 0$ and double roots at $x = \pm 2$. So, the curve crosses the x axis at the origin and is tangent to it at $x = \pm 2$:



2614. Applying g^{-1} to equation ② and h to ③,

$$\begin{aligned} \mathrm{h}(0) &= \mathrm{g}^{-1}(a), \\ \mathrm{f}^{-1}(a) &= \mathrm{h}(0). \end{aligned}$$

Hence, $\mathrm{g}^{-1}(a) = \mathrm{f}^{-1}(a)$. Substituting this into ①, $\mathrm{g}\mathrm{g}^{-1}(a) = 0$, so $a = 0$, as required.

2615. (a) $\log_x\left(y^2\right) \equiv 2\log_x y$,

(b) $\log_{x^2}\left(y^2\right) \equiv \log_x y$,

(c) $\log_{x^2} y \equiv \log_x y^{\frac{1}{2}} \equiv \frac{1}{2}\log_x y$.

2616. Let $u = 2 + \sqrt{x}$. This gives $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$. So,

$$dx = 2\sqrt{x}du = 2(u-2)du.$$

We can now substitute:

$$\begin{aligned} &\int_{x=0}^{x=1} \frac{2}{2+\sqrt{x}}\, dx \\ = &\int_{u=2}^{u=3} \frac{2}{u} \cdot 2(u-2)\, du. \end{aligned}$$

Simplifying the integrand, this is

$$\begin{aligned} &4\int_2^3 1-\tfrac{2}{u}\,du \\ &= 4\Big[u-2\ln|u|\Big]_2^3 \\ &= 4(3-2\ln 3-2+2\ln 2) \\ &= 4+8(\ln 2-\ln 3) \\ &= 4+8\ln\tfrac{2}{3},\text{ as required.}\end{aligned}$$

2617. The unit circle is $x^2+y^2=1$, meaning we need

$$\sin^2 t+\sin^2 2t=1.$$

Using a double-angle formula, this is

$$\sin^2 t+4\sin^2 t\cos^2 t=1.$$

Substituting $\cos^2 t\equiv 1-\sin^2 t$ gives a quadratic in $\sin^2 t$:

$$\begin{aligned} &\sin^2 t+4\sin^2 t(1-\sin^2 t)=1 \\ \Longrightarrow\ &4\sin^4 t-5\sin^2 t+1=0 \\ \Longrightarrow\ &(4\sin^2 t-1)(\sin^2 t-1)=0 \\ \Longrightarrow\ &\sin t=\pm\tfrac{1}{2},\pm 1 \\ \therefore\ \ &t=\pm\tfrac{\pi}{6},\pm\tfrac{5\pi}{6},\pm\tfrac{\pi}{2}.\end{aligned}$$

There are six intersections for $t\in(-\pi,\pi]$. These are $(\pm 1,0)$ and the four points $(\pm 1/2,\pm\sqrt{3}/2)$.

2618. The conditional probability formula gives

$$\begin{aligned} &\mathbb{P}(A\mid B)=\mathbb{P}(B\mid A) \\ \Longleftrightarrow\ &\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}=\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)}.\end{aligned}$$

$\mathbb{P}(A\cap B)\neq 0$, since A,B aren't mutually exclusive. Hence, we can divide through by it, maintaining implication in both directions:

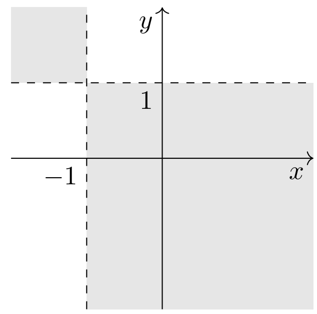
$$\begin{aligned} &\frac{1}{\mathbb{P}(B)}=\frac{1}{\mathbb{P}(A)} \\ \Longleftrightarrow\ &\mathbb{P}(A)=\mathbb{P}(B),\text{ as required.}\end{aligned}$$

2619. The gradient is given by $m=\frac{\Delta y}{\Delta x}$. Using the first Pythagorean trig identity,

$$\begin{aligned} m&=\frac{1-\cos^2 a}{\sin^2 a-1} \\ &\equiv\frac{\sin^2 a}{-\cos^2 a} \\ &\equiv-\tan^2 a,\text{ as required.}\end{aligned}$$

2620. The distribution of X, defined as the number of fives showing on the six dice, is $X\sim B(6,1/6)$. So, $\mathbb{P}(X=0)\approx 0.33$ and $\mathbb{P}(X=1)\approx 0.40$. Hence, one five is more probable.

2621. Factorising, we have $(x+1)(y-1)<0$. The LHS is negative iff one of the factors is negative and one is positive. This gives



2622. (a) Splitting the fraction up,

$$\frac{12}{5}=\frac{2\times 5+2}{5}=2+\frac{2}{5}.$$

(b) Applying the same technique,

$$\begin{aligned} &\frac{4x^2+x}{x+1} \\ \equiv\ &\frac{(4x-3)(x+1)+3}{x+1} \\ \equiv\ &4x-3+\frac{3}{x+1}.\end{aligned}$$

2623. The equation $y=\mathrm{g}(x)$ is a line. The signed area from $-a$ to a is zero, so this line must pass through the origin, with equation $y=mx$ for some m. The second integral is then

$$\begin{aligned} &\int_a^{3a} mx\,dx \\ \equiv\ &\Big[\tfrac{1}{2}mx^2\Big]_a^{3a} \\ \equiv\ &\tfrac{9}{2}a^2m-\tfrac{1}{2}a^2m \\ \equiv\ &4a^2m.\end{aligned}$$

So, $4a^2m=4a$. Since $a\neq 0$, this gives $m=\frac{1}{a}$. Hence, $\mathrm{g}(a)=ma=\frac{1}{a}\cdot a=1$.

2624. The possibility space consists of 24 outcomes. Counting successful ones, the distribution of X is

x	0	1	2	3	4	5
$\mathbb{P}(X=x)$	$\frac{4}{24}$	$\frac{7}{24}$	$\frac{6}{24}$	$\frac{4}{24}$	$\frac{2}{24}$	$\frac{1}{24}$

The probability that two successive simultaneous rolls yield the same value of X is the sum of the squares of these probabilities:

$$\begin{aligned} p&=\tfrac{1}{24^2}\left(4^2+7^2+6^2+4^2+2^2+1^2\right) \\ &=\tfrac{61}{288}.\end{aligned}$$

2625. We enact the operators, and rearrange:

$$\begin{aligned}&\frac{d}{dx}\left(x+\frac{d}{dx}(x+y)\right)=x\\ \Longrightarrow\ &\frac{d}{dx}\left(x+1+\frac{dy}{dx}\right)=x\\ \Longrightarrow\ &1+\frac{d^2y}{dx^2}=x\\ \Longrightarrow\ &\frac{d^2y}{dx^2}=x-1.\end{aligned}$$

2626. (a) These are basic freefall acceleration and an air resistance term.

(b) Separating the variables,

$$\begin{aligned}&\frac{1}{g-0.1v}\frac{dv}{dt}=1\\ \Longrightarrow\ &\int\frac{1}{g-0.1v}\,dv=\int 1\,dt\\ \Longrightarrow\ &-10\ln|g-0.1v|=t+c.\end{aligned}$$

The skydiver begins at (vertical) rest, so $v=0$ when $t=0$. Hence, $c=-10\ln g$. We can drop the mod signs, because $g-0.1v$ is acceleration, which must be positive in this scenario.

$$\begin{aligned}&-10\ln(g-0.1v)=t-10\ln g\\ \Longrightarrow\ &\ln(g-0.1v)=-0.1t+\ln g\\ \Longrightarrow\ &g-0.1v=ge^{-0.1t}\\ \Longrightarrow\ &v=10g-10ge^{-0.1t}\\ \Longrightarrow\ &v=98-98e^{-0.1t},\quad\text{as required.}\end{aligned}$$

(c) Terminal velocity is the limit as $t\to\infty$; which is $v\to 98$. Setting $v=49$,

$$\begin{aligned}&98-98e^{-0.1t}=49\\ \Longrightarrow\ &e^{-0.1t}=\tfrac{1}{2}\\ \Longrightarrow\ &t=10\ln 2\text{ seconds.}\end{aligned}$$

2627. (a) False. The function f' is the same in each case, but the inputs are different. If $f'(x)=x^3$, then the two sides are x^3 and $(x+2)^3$.

(b) False. This is a restatement of (a).

(c) True. By the chain rule,

$$\frac{d}{dx}f(u)=f'(u)\times\frac{du}{dx}.$$

And $u=x+2$, so $\frac{du}{dx}=1$.

2628. The logarithms $\log_p q$ and $\log_q p$ are reciprocals of each other. This produces a quadratic in $\log_2 x$:

$$\begin{aligned}&32\log_x 2+\log_2 x=12\\ \Longrightarrow\ &\frac{32}{\log_2 x}+\log_2 x=12\\ \Longrightarrow\ &(\log_2 x)^2-12\log_2 x+32=0\\ \Longrightarrow\ &(\log_2 x-4)(\log_2 x-8)=0\\ \Longrightarrow\ &\log_2 x=4,8\\ \Longrightarrow\ &x=16,256.\end{aligned}$$

2629. (a) For the tension in the string to be modelled as constant, we assume the string is light.

(b) Assuming the 3 and 4 kg blocks slide as one, NII for the system is $5g=12a$, so $a=\frac{5}{12}g$.

At this acceleration, the horizontal equation of motion for the 4 kg block is $F=ma=\frac{5}{3}g$, and the reaction force on the 4 kg block is $R=4g$:

$$F_{\max}=\tfrac{1}{2}\cdot 4g=2g>\tfrac{5}{3}g.$$

Hence, the friction between the two blocks is indeed big enough to guarantee that they slide as one, as assumed. The friction between them will be $\frac{5}{3}g$ N.

2630. (a) The roots of $ax^2+bx-a^3=0$ must be $x=\pm 1$, as these are the boundaries of the solution set of the inequality. Hence, the parabola $y=ax^2+bx-a^3$ must be symmetrical about the y axis, so $b=0$.

(b) $ax^2-a^3=0$ must have roots $x=\pm 1$, giving $a^3/a=1$. Hence, $a=\pm 1$. But, for the solution set to lie *between* -1 and 1, the parabola must be positive. So, $a=1$.

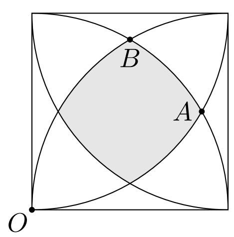
2631. Visualise the (x,y) plane as horizontal, and the z axis as vertical.

In the (x,y) plane, $r=\sqrt{x^2+y^2}$ is the distance from O. Hence, we are setting $z=r$, i.e. setting the height above the horizontal plane equal to the (x,y) radius. From O, this gives a circle whose radius increases at a constant rate with z. This is an inverted cone with semi-vertical angle 45° and vertex at the origin.

2632. The boundary equations have solutions $x=\pm 2$ and $y=-\frac{1}{2}(1\pm\sqrt{17})\approx -2.56, 1.56$. So, the first set is $\{-1,0,1\}$; the second set is $\{-2,-1,0,1\}$. Hence, the intersection is $\{-1,0,1\}$.

2633. The sum isn't binomially distributed. The random variables $X_1\sim B(1,0.001)$ and $X_2\sim B(1,0.999)$ provide a counterexample: the sum X_1+X_2 takes values $\{0,1,2\}$ symmetrically around 1, but the outcome 1 is almost certain. There is no binomial distribution $B(2,p)$ with such probabilities.

2634. Place the square at O, and name two intersections as shown.

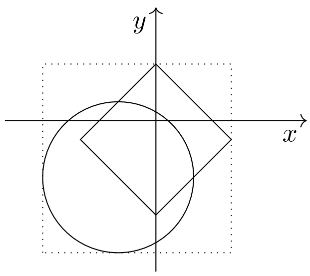


Point A has $y = \frac{1}{2} = \sin 30°$, so $x = \cos 30° = \frac{\sqrt{3}}{2}$. The angle subtended at O by arc AB is therefore $30°$. So, sector OAB has area $\frac{\pi}{12}$, and $\triangle OAB$ has area $\frac{1}{2}\sin 30° = \frac{1}{4}$. Hence, the area of segment AB is $\frac{\pi}{12} - \frac{1}{4}$.

The shaded area then consists of four copies of this segment, plus a square. The square has diagonal length $1 - 2(1 - \cos 30°) = \sqrt{3} - 1$, so its area is $\frac{1}{2}(\sqrt{3}-1)^2 = 2 - \sqrt{3}$. The shaded area is therefore

$$\begin{aligned} A &= 4 \times \left(\tfrac{\pi}{12} - \tfrac{1}{4}\right) + 2 - \sqrt{3} \\ &= \tfrac{\pi}{3} + 1 - \sqrt{3}, \text{ as required.} \end{aligned}$$

2635. The first curve is a circle: $(x+2)^2 + (y+3)^2 = 16$ has centre $(-2, -3)$ and radius 4. The second curve is a square, centred on $(0, -1)$, with vertices at $(4, -1)$ and $(0, 3)$.



The minimum bounding rectangle, therefore, has sides $x = 4, -6$ and $y = 3, -7$, as shown. So, it is a square with side length 10.

2636. Solving for intersections,

$$\begin{aligned} x &= \frac{1}{x-1} + \frac{1}{(x-1)^2} \\ \Longrightarrow x(x-1)^2 &= (x-1) + 1 \\ \Longrightarrow x^3 - 2x^2 &= 0 \\ \Longrightarrow x^2(x-2) &= 0. \end{aligned}$$

This has a double root at $x = 0$, which means the curve and the line must have a point of tangency at the origin.

2637. (a) There are four ways of choosing a set of three suits. Having chosen, there are 13^3 ways of picking a card from each suit. This gives $4 \times 13^3 = 8788$.

(b) $13^3 = 2197$.

(c) $\mathbb{P}(\text{no spades} \mid \text{all different}) = \frac{2197}{8788} = \frac{1}{4}$.

(d) Given three different suits, exactly one suit is missing. This is equally likely to be any suit. Hence, without using the calculations in (a), (b) and (c), the probability that it is spades that is the missing suit is $\frac{1}{4}$.

2638. For f to be polynomial, the denominator would have to be a factor of the numerator. Attempting to factorise, we would need

$$\begin{aligned} &6x^4 + x^3 + 4x^2 + x - 4 \\ &\equiv (x^2+1)(ax^2 + bx + c). \end{aligned}$$

The term in x^4 requires $a = 6$. Then the term in x^3 needs $b = 1$. Then the term in x^2 needs $c = -2$. This gives

$$\begin{aligned} &6x^2 + x^3 + 4x^2 + x - 4 \\ &\equiv (x^2+1)(6x^2 + x - 2) - 2. \end{aligned}$$

So, $x^2 + 1$ is not a factor, and the quotient cannot be expressed as a polynomial in x. Hence, f is not a polynomial function.

2639. (a) The derivative is $\frac{dy}{dx} = 2x$, so the gradient of the normal is $m = {-1}/{2p}$ at (p, p^2). The normal has equation

$$\begin{aligned} y - p^2 &= {-1}/{2p}(x - p) \\ \Longrightarrow 2py &= 2p^3 + p - x, \text{ as required.} \end{aligned}$$

(b) Substituting $(3, 0)$, we get $0 = 2p^3 + p - 3$. With a polynomial solver, $p = 1$. The point $(3, 0)$ is on the normal through $(1, 1)$, at distance $\sqrt{5}$. Substituting $(0, 4)$, we get $8p = 2p^3 + p$. So, $p = 0, \pm\sqrt{3.5}$. $(0, 4)$ is on the normal through $(0, 0)$, distance 4, and through $(\pm\sqrt{3.5}, 3.5)$, distance 1.94 (3sf).
Since $1.94 < \sqrt{5}$, the point $(0, 4)$ is closer.

2640. Assume, for a contradiction, that the third vertex lies at (p, q), where q is an integer. Equating the lengths of the sides, we know that $\sqrt{p^2 + q^2} = 2p$, which gives $p^2 + q^2 = 4p^2$, so $q^2 = 3p^2$. Hence, $\sqrt{3} = q/p$, where p and q are integers, which means $\sqrt{3}$ is rational. This is a contradiction. Hence, the y coordinate $q \notin \mathbb{Z}$. $\square$

2641. The derivative of $\tan x$ is $\sec^2 x$. Using the reverse chain rule,

$$\int \sec^2 \tfrac{1}{3}x \, dx = 3\tan \tfrac{1}{3}x + c.$$

2642. (a) In the limit $\varepsilon \to 0$, we have $-2x + 10 = 0$. This equation has a root at $x = 5$.

(b) The quadratic formula gives

$$x = \frac{2 \pm \sqrt{4 - 40\varepsilon}}{2\varepsilon} \equiv \frac{1 \pm \sqrt{1 - 10\varepsilon}}{\varepsilon}.$$

(c) For small ε, the (positive) numerator tends to $1 + \sqrt{1} = 2$, giving a root near $x = \frac{2}{\varepsilon}$.

(d) Taking the negative root, the numerator does tend to zero. But so does the denominator. In such cases, further analysis is needed to work out the limit. In this case, we know from (a) that the limit must, in fact, be $x = 5$.

2643. This is true. It is an instance of the chain rule:

$$\frac{d(\cos 2\theta)}{dt} \times \frac{dt}{d\theta} \equiv \frac{d(\cos 2\theta)}{d\theta} \equiv -2\sin 2\theta.$$

— Nota Bene —

It is not incorrect, although it must be treated with considerable care, to think of the instances of dt cancelling in the above. What is actually going on is that the finite δt's which are implicit in the derivatives (prior to taking the infinitesimal limit $\delta t \to 0$) cancel. This is why Leibniz's d notation is so useful: the derivatives do act like fractions.

2644. (a) The area from $x = 2$ to $x = p$ is given by

$$\begin{aligned} \text{A}(p) &= \int_2^p 12x^3 + 10x + 2\,dx \\ &\equiv \Big[3x^4 + 5x^2 + 2x\Big]_2^p \\ &\equiv 3p^4 + 5p^2 + 2p - 72. \end{aligned}$$

(b) Setting $\text{A}(p) = 7390$, we rearrange to

$$3p^4 + 5p^2 + 2p - 7462 = 0.$$

A quartic solver gives a negative root, which we reject as per the question, and $p = 7$.

2645. (a) Solving simultaneously, we add the equations to get $2(x-a)^2 = 2r^2$. This gives $x = a \pm r$ and $y = b$.

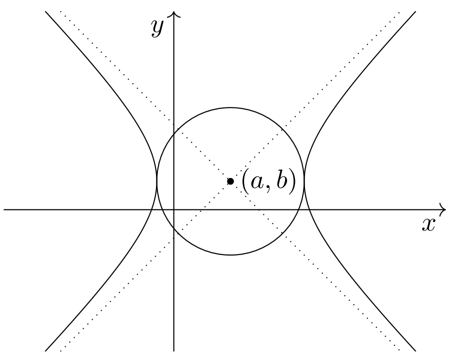
(b) Differentiating the hyperbola implicitly,

$$\begin{aligned} &(x-a)^2 - (y-b)^2 = r^2 \\ \Longrightarrow\ & 2(x-a) - 2(y-b)\frac{dy}{dx} = 0 \\ \Longrightarrow\ & \frac{dy}{dx} = \frac{x-a}{y-b}. \end{aligned}$$

At $y = b$, the gradient is undefined, so the tangents are parallel to the y axis. They have equations $x = a \pm r$.

(c) Both points of intersection lie $y = b$, which is a diameter of the circle. Hence, since tangent is perpendicular to radius, the tangents to the circle are also $x = a \pm r$. Therefore, the curves are tangent to each other.

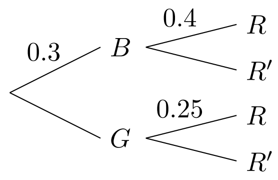
(d) Both curves are centred on the point (a, b):



2646. (a) This is false. Solving $e^x - 2 = 0$, we get $x = \ln 2$ as a counterexample.

(b) This is true, because $e^{x-2} = 0$ has no roots.

2647. (a) Conditioning on bogus/genuine, we have



(b) $\mathbb{P}(R) = 0.3 \times 0.4 + 0.7 \times 0.25 = 0.295.$

(c) Restricting the possibility space to the two branches above,

$$\begin{aligned} \mathbb{P}(G \mid R) &= \frac{\mathbb{P}(G \cap R)}{\mathbb{P}(R)} \\ &= \frac{0.7 \times 0.25}{0.3 \times 0.4 + 0.7 \times 0.25} \\ &= 0.593 \text{ (3sf)}. \end{aligned}$$

2648. An equilateral triangle of side length 1 has height $\sqrt{3}/2$. The centre of the triangle divides this height in the ratio 1 : 2. Hence, the largest radius which can rotate inside the triangle has the following length, which is the distance between the centre of the triangle and the midpoints of its sides:

$$r = \tfrac{1}{3} \times \tfrac{\sqrt{3}}{2} = \tfrac{\sqrt{3}}{6}.$$

Calling the side length of the square l, we set this radius equal to half the diagonal of the square:

$$\begin{aligned} &\tfrac{\sqrt{3}}{6} = \tfrac{\sqrt{2}}{2}l \\ \Longrightarrow\ & \tfrac{1}{12} = \tfrac{1}{2}l^2 \\ \Longrightarrow\ & l^2 = \tfrac{1}{6} \\ \therefore\ \ & A = \tfrac{1}{6}. \end{aligned}$$

We require no intersections, so this upper bound is not attainable. And we also require the square to have non-zero size, so the lower bound $A = 0$ is not attainable either. Hence, the area of the square satisfies $A \in (0, 1/6)$, as required.

2649. The perpendicular bisector of $(5, 1)$ and $(1, 1)$ is $x = 3$, and of $(4, 4)$ and $(1, 1)$ is $y = 5 - x$. This gives the proposed centre as $(3, 2)$, and radius as $\sqrt{5}$. Checking the point $(2, 0)$, we see that it also lies at a distance $\sqrt{5}$ from $(3, 2)$. So, all four points lie on the same circle. □

2650. Taking natural logs, we simplify with log rules:

$$\begin{aligned} &y = a \cdot b^x \\ \Longrightarrow\ & \ln y = \ln(a \cdot b^x) \\ \Longrightarrow\ & \ln y = \ln a + x \ln b. \end{aligned}$$

Since $\ln a$ and $\ln b$ are constants, this is a linear relationship between $\ln y$ and x.

2651. (a) Transforming from $y = \mathrm{f}(x)$ to $y = \mathrm{f}\big(\frac{1}{2}x\big)$ is a stretch, scale factor 2, in the x direction. This matches the scaled limits, so the integral has value $2b$.

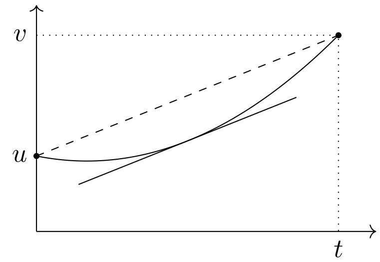
(b) From $y = \mathrm{f}(x)$ to $y = \mathrm{f}\big(\frac{1}{2}x + a\big)$ is a stretch, scale factor 2, in the x direction followed by a translation by $-a$ in the x direction. Again, this matches the limits. Translation doesn't affect the signed area, which is again $2b$.

2652. Completing the square for x and y,

$$\begin{aligned} & 9x^2 - 6x + y^2 + 8y + 7 = 0 \\ \Longrightarrow\ & 9\big(x - \tfrac{1}{3}\big)^2 + (y+4)^2 = 10. \end{aligned}$$

So, the ellipse has centre $\big(1/3, -4\big)$.

2653. A velocity time graph is as follows:



The average velocity is the gradient of the dashed chord; the instantaneous velocity is the gradient of the solid tangent. At the point where the curve departs maximally from the dashed chord, the two gradients must be equal.

— Nota Bene —

This visually obvious idea has a formal name in mathematics: Lagrange's *mean value theorem*. Like the factor theorem or Pythagoras's theorem, it is a simple result with widespread application in more complicated proofs.

2654. Multiplying up by the denominators,

$$2 \equiv P(x^3+1) + Qx^3.$$

Equating coefficients of x^3, $0 = P + Q$; equating the constant terms, $2 = P$. So, the required values are $P = 2$ and $Q = -2$.

2655. This isn't true, due to the constant of integration. Integrating the first equation, if $\mathrm{f}'(x) = \mathrm{f}(x) + c$, then the LHS and RHS of the second statement may differ by a constant.

2656. The function $x \mapsto \sin x$ must be restricted before it can be inverted. Hence, the points on the graph $y = \sin x$ which satisfy $x = \arcsin y$ are a subset of those which satisfy $y = \sin x$. So, the implication is $y = \sin x \impliedby x = \arcsin y$.

2657. This is a quadratic in $\tan x$. Using the formula,

$$\begin{aligned} & \sqrt{3}\tan^2 x - 4\tan x + \sqrt{3} = 0 \\ \Longrightarrow\ & \tan x = \sqrt{3}, \tfrac{1}{\sqrt{3}} \\ \therefore\ \ & x = \tfrac{\pi}{6}, \tfrac{\pi}{3}, \tfrac{7\pi}{6}, \tfrac{4\pi}{3}. \end{aligned}$$

2658. The initial velocities are $\frac{1}{2}u$ vertically and $\frac{\sqrt{3}}{2}u$ horizontally. Using these, the vertical *suvat* is

$$-\sqrt{3} = \tfrac{1}{2}ut - \tfrac{1}{2}gt^2.$$

The horizontal is

$$\tfrac{\sqrt{3}}{2}ut = 5 \implies t = \tfrac{10}{u\sqrt{3}}.$$

We can now substitute for t:

$$\begin{aligned} & -\sqrt{3} = \tfrac{1}{2}u\left(\tfrac{10}{u\sqrt{3}}\right) - \tfrac{1}{2}g\left(\tfrac{10}{u\sqrt{3}}\right)^2 \\ \Longrightarrow\ & -\sqrt{3} = \tfrac{5\sqrt{3}}{3} - \tfrac{50g}{3u^2} \\ \Longrightarrow\ & \tfrac{50g}{3u^2} = \tfrac{8\sqrt{3}}{3} \\ \Longrightarrow\ & u^2 = \tfrac{25g}{4\sqrt{3}}, \text{ as required.} \end{aligned}$$

2659. The two constituent transformations are reflection in the y axis, followed by translation by vector $k\mathbf{i}$. The combined effect is reflection in the line $x = \frac{k}{2}$.

2660. The derivative is $2x + k$. So, the equation of the tangent at $(p, p^2 + kp)$ is

$$\begin{aligned} & y - (p^2 + kp) = (2p+k)(x-p) \\ \Longrightarrow\ & y = (k+2p)x - p^2. \end{aligned}$$

Likewise at $x = q$. Solving simultaneously for the x coordinate of the intersection of the tangents,

$$\begin{aligned} & (k+2p)x - p^2 = (k+2q)x - q^2 \\ \Longrightarrow\ & kx + 2px - p^2 = kx + 2kq - q^2 \\ \Longrightarrow\ & 2px - 2qx = p^2 - q^2 \\ \Longrightarrow\ & 2(p-q)x = (p-q)(p+q). \end{aligned}$$

Since $x = p$ and $x = q$ are distinct, $p - q \neq 0$. So, we can divide through by $(p-q)$, which leaves $x = \frac{1}{2}(p+q)$, as required.

2661. (a) A four-strip trapezium rule has strips of width $\frac{1}{4}$. Defining $\mathrm{f}(x) = e^x\sqrt{x}$, I is approximately

$$\tfrac{1}{8}\big(\mathrm{f}(1) + 2(\mathrm{f}(1.25) + \mathrm{f}(1.5) + \mathrm{f}(1.75)) + \mathrm{f}(2)\big).$$

Evaluating, $I \approx 5.897$ (4sf).

(b) Differentiating twice,

$$\begin{aligned} & \mathrm{f}(x) = e^x x^{\frac{1}{2}} \\ \Longrightarrow\ & \mathrm{f}'(x) = e^x\big(x^{\frac{1}{2}} + \tfrac{1}{2}x^{-\frac{1}{2}}\big) \\ \Longrightarrow\ & \mathrm{f}''(x) = e^x\big(x^{\frac{1}{2}} + x^{-\frac{1}{2}} - \tfrac{1}{4}x^{-\frac{3}{2}}\big). \end{aligned}$$

This gives $f''(1) \approx 4.8 > 0$. Since there are no points of inflection on $[1, 2]$, nor any points where the second derivative is undefined, the second derivative must be positive everywhere on $[1, 2]$. This means that the trapezium rule will overestimate the integral.

2662. Suppose that parabolae $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ have three distinct points in common. Then the equation $\mathrm{f}(x) = \mathrm{g}(x)$ has three distinct roots. This equation has degree at most two, so $\mathrm{f}(x)$ and $\mathrm{g}(x)$ must be identical. Hence, if two such parabolae have three distinct points in common, then they must be the same parabola. QED.

2663. The expectation is $\mathbb{E}(X) = 6 \times {}^1\!/_4 = 1.5$. So, we consider outcomes 1 and 2. With $X \sim B(6, {}^1\!/_4)$,

$$\mathbb{P}(X=1) = {}^6\mathrm{C}_1\left(\tfrac{1}{4}\right)^1\left(\tfrac{3}{4}\right)^5 \approx 0.356,$$
$$\mathbb{P}(X=2) = {}^6\mathrm{C}_2\left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right)^4 \approx 0.297.$$

So, the mode is 1.

2664. The curve crosses the x axis at

$$x = -\tfrac{1}{2}, -\tfrac{3}{2}, -\tfrac{5}{2}.$$

Because these roots are in AP, the cubic's centre of rotational symmetry must be at $(-{}^3\!/_2, 0)$. Hence, the two regions enclosed must be rotated images of one another. So, their areas are equal, as required.

2665. Ignoring leap years etc., the probability that any individual has a birthday in a given week is $\frac{7}{365}$. So, with n people, the probability that at least one has a birthday is given by

$$\begin{aligned}\mathbb{P}(\text{at least one}) &= 1 - \mathbb{P}(\text{none})\\ &= 1 - \left(\tfrac{358}{365}\right)^n.\end{aligned}$$

Equating this to ${}^1\!/_2$, we get $n = \log_{\frac{358}{365}} {}^1\!/_2 \approx 35.8$. So, 36 people are required.

2666. This is a quadratic in 2^x:

$$\begin{aligned}&2^{3+2x} + 2^{1+x} - 1 = 0\\ \Longrightarrow\ &8 \cdot (2^x)^2 + 2 \cdot (2^x) - 1 = 0\\ \Longrightarrow\ &(4 \cdot 2^x - 1)(2 \cdot 2^x + 1) = 0\\ \Longrightarrow\ &2^x = -1, \tfrac{1}{4}.\end{aligned}$$

The former equation has no roots, so $x = -2$.

2667. (a) Using the function facility on a calculator, the predicted and observed efficiencies are

$T°$	60	65	70	75	80
E_{O}	20	39	71	76	21
E_{P}	20	38.75	70	76.25	20

These are in close agreement, so the model is consistent with the data.

(b) The model is a cubic.

$$\begin{aligned}&E = -0.05T^3 + 10T^2 - 660T + 14420\\ \Longrightarrow\ &\frac{dE}{dT} = -0.15T^2 + 20T - 660.\end{aligned}$$

Setting this to zero for SPs, $T = 60°, 73.\dot{3}°$. E is a negative cubic, so these are min and max respectively. So, optimum is $T = 73.\dot{3}°$.

2668. (a) Since the cubic has precisely two roots, one of them must be a double root. This means a point of tangency with the x axis, which must be a stationary point.

(b) For SPs,

$$\begin{aligned}&225x^2 - 70x - 8 = 0\\ \Longrightarrow\ &x = -{}^4\!/_{45}, {}^2\!/_5.\end{aligned}$$

Checking these values, the relevant SP is at $({}^2\!/_5, 0)$. So, by the factor theorem, $(5x-2)$ is a repeated factor of the cubic. Taking it out, $(5x-2)^2(3x+1) = 0$, so $x = {}^2\!/_5, -{}^1\!/_3$.

2669. In each case, we find the range of the denominator first, then reciprocate it:

(a) $[1, \infty)$, so $(0, 1]$,

(b) $(-\infty, \infty)$, so $\mathbb{R} \setminus \{0\}$,

(c) $[1, \infty)$, so $(0, 1]$.

2670. (a) Adding the first and third branches,

$$\mathbb{P}(B) = \tfrac{1}{4} \times \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{5}{16}.$$

(b) Restricting the possibility space,

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\frac{1}{16}}{\frac{5}{16}} = \tfrac{1}{5}.$$

2671. (a) Taking out x^2 inside the square root, and using the fact that $x > 0$,

$$\sqrt{x^2 + p} \equiv \sqrt{x^2\left(1 + \frac{p}{x^2}\right)} \equiv x\left(1 + \frac{p}{x^2}\right)^{\frac{1}{2}}.$$

(b) The generalised binomial expansion gives

$$\begin{aligned}&\sqrt{1 + \tfrac{p}{x^2}}\\ &\equiv 1 + \tfrac{1}{2}\left(\tfrac{p}{x^2}\right) + \frac{{}^1\!/_2 \cdot -{}^1\!/_2}{2!}\left(\tfrac{p}{x^2}\right)^2 + \ldots\\ &\equiv 1 + \frac{p}{2x^2} - \frac{p^2}{8x^4} + \ldots\end{aligned}$$

(c) Using these results, the limit L is

$$\begin{aligned}&\lim_{p \to 0} \frac{x\left(1 + \frac{p}{2x^2} - \frac{p^2}{8x^4} + \ldots\right) - x}{p}\\ &= \lim_{p \to 0} \frac{\frac{p}{2x^2} - \frac{p^2}{8x^4} + \ldots}{p}\\ &= \lim_{p \to 0} \frac{1}{2x^2} - \frac{p}{8x^4} + \ldots\end{aligned}$$

At this point we can take the limit. Since all terms beyond the first contain factors of p,

$$L = \frac{1}{2x^2}.$$

2672. Rearranging the identity to $\cos^2 x \equiv \frac{1}{2}(\cos 2x + 1)$,

$$\begin{aligned}&\int \cos^2\left(4x + \tfrac{\pi}{12}\right)\,dx\\ &= \tfrac{1}{2}\int \cos\left(8x + \tfrac{\pi}{6}\right) + 1\,dx\\ &= \tfrac{1}{16}\sin\left(8x + \tfrac{\pi}{6}\right) + \tfrac{1}{2}x + c.\end{aligned}$$

2673. For positive x, these produce equivalent outputs, according to the relevant log rule:

$$\log_a b^n \equiv n\log_a b.$$

However, the (broadest possible) domain of f is $\mathbb{R}\setminus\{0\}$, whereas the (broadest possible) domain of g is $\mathbb{R}^+$. Hence, e.g. $\mathrm{f}(-10) = 2$, but $\mathrm{g}(-10)$ is undefined.

— Nota Bene —

This doesn't break the log rule $\log_a b^n \equiv n\log_a b$. If no domains are mentioned, such an identity should be considered as coming with an implicit assumption that both sides are well defined.

2674. The curve passes through O, which means that k would have to be $\sqrt{2}$. The function

$$\mathrm{f}(x) = \sqrt{x^3+2} - \sqrt{2}$$

has domain $\left[\sqrt[3]{-2}, \infty\right)$ and range $\left[-\sqrt{2}, \infty\right)$, both of which are consistent with the graph. Also, the derivative is

$$\mathrm{f}'(x) = x^2\left(x^3+2\right)^{-\frac{1}{2}}.$$

This is undefined at $x = \sqrt[3]{-2}$, giving a tangent parallel to y, as in the diagram. There is also a single SP at $(0,0)$, as in the diagram. Hence, yes, the equation $y = \sqrt{x^3+2} - \sqrt{2}$ could generate the graph.

— Nota Bene —

You might also consider the second derivative, which should show a point of inflection at O. By the quotient rule,

$$\frac{d^2y}{dx^2} = \frac{3x(x^3+8)}{4(x^3+2)^{\frac{3}{2}}}.$$

The numerator has a single factor of x. Hence, the second derivative is zero and changes sign at the origin. So, the equation has a point of inflection at O, which matches the graph shown.

2675. Let $z = e^{2x} - 1$. This gives

$$\begin{aligned}&z^4 - z^2 = 0\\ \Longrightarrow\ &z^2(z+1)(z-1) = 0\\ \Longrightarrow\ &z = -1, 0, 1.\end{aligned}$$

Hence, $e^{2x} - 1 = -1, 0, 1$, giving $e^{2x} = 0, 1, 2$. The first of these has no roots. The second and third yield roots $x = 0$ and $x = \frac{1}{2}\ln 2$.

2676. Since $3P = 2P + P$, it is true that the object can only remain in equilibrium if the forces have the same line of action. However, this is not sufficient. If all three act in the same direction along the same line of action, then the object will accelerate.

2677. Let the lengths be $1, r, r^2$, with $r > 1$. Then, by Pythagoras, $1 + r^2 = r^4$, so $r^4 - r^2 - 1 = 0$. This is a quadratic in r^2. The formula gives $r^2 = \phi$ or a negative root, which we reject since $r^2 \geq 0$. We then take the positive root, since $r > 1$. This gives $r = \sqrt{\phi}$, as required.

2678. Substituting into the LHS,

$$\begin{aligned}&x^2(y^2-1)\\ \equiv\ &\sec^2 t\big((\sin t + \cos t)^2 - 1\big)\\ \equiv\ &\sec^2 t(\sin^2 t + \cos^2 t + 2\sin t\cos t - 1)\\ \equiv\ &\sec^2 t(2\sin t\cos t)\\ \equiv\ &2\tan t.\end{aligned}$$

Substituting into the RHS,

$$\begin{aligned}&2(xy-1)\\ \equiv\ &2\big(\sec t(\sin t + \cos t) - 1\big)\\ \equiv\ &2(\tan t + 1 - 1)\\ \equiv\ &2\tan t.\end{aligned}$$

Since LHS $\equiv$ RHS, the result holds.

2679. (a) Quoting a standard result, the sample mean is distributed $\bar{X} \sim N(\mu, \sigma^2/n)$.

(b) Translation affects mean but not variance, so $X_i + 2 \sim N(\mu + 2, \sigma^2)$.

(c) This is $n\bar{X}$. So, we scale the result from (a): the mean by n and the variance by n^2. Hence, the distribution is $N(n\mu, n\sigma^2)$.

2680. (a) The parabolae are reflections in $y = x$. So, we solve $y = x$ and $y = x^2 - x$ simultaneously:

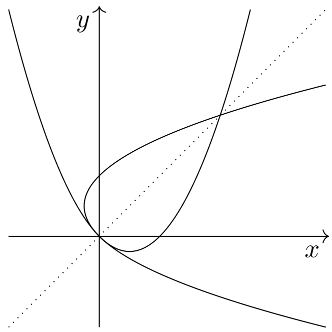
$$\begin{aligned}&x = x^2 - x\\ \Longrightarrow\ &x^2 - 2x = 0\\ \Longrightarrow\ &x = 0, 2.\end{aligned}$$

The derivative for the first parabola is

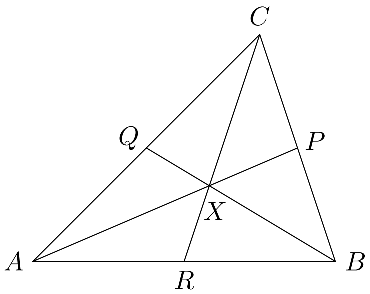
$$\frac{dy}{dx} = 2x - 1.$$

So, the gradient at $x = 0$ is -1 and $x = 2$ is 3. Hence, at $x = 0$ the first parabola is normal to $y = x$. So, the second parabola must have the same gradient at this point, which means the parabolae are tangent at the origin.

(b) With $y = x$ drawn dotted, the sketch is



2681. Labelling points, the scenario is



$\triangle ARX$ and $\triangle BRX$ have the same area, because $AR = RB$ and they also have the same height. The same is true for $\triangle AQX$ and $\triangle CQX$.

Since the centroid X cuts the median RC in the ratio $1 : 2$, $\triangle ARC$ has three times the area of $\triangle ARX$. So, $\triangle AXC$ has twice the area of $\triangle ARX$. Hence, $\triangle AQX$ and $\triangle CQX$ have the same area as $\triangle ARX$ and $\triangle ABX$.

The same argument can then be applied to $\triangle BPX$ and $\triangle CPX$. So, the six triangles formed by the medians all have the same area. □

2682. In fact, both students are right and wrong.

Neither answer contains a constant of integration, which is where the answer lies. A log rule gives

$$\ln|2x| \equiv \ln 2 + \ln|x|,$$

which means the students' answers only differ by a constant. Hence, if the answers are instead given as $I = \frac{1}{2}\ln|x| + c$ and $I = \frac{1}{2}\ln|2x| + d$, then they are equally correct.

2683. Let $\text{f}(x) = \sin^2 x + \sin x + 1$. For SPs in $[0, 2\pi)$,

$$\begin{aligned} & 2\sin x\cos x + \cos x = 0 \\ \Longrightarrow\ & \cos x(2\sin x + 1) = 0 \\ \Longrightarrow\ & \cos x = 0 \text{ or } \sin x = -\tfrac{1}{2} \\ \Longrightarrow\ & x = \tfrac{\pi}{2}, \tfrac{3\pi}{2}, \tfrac{7\pi}{6}, \tfrac{11\pi}{6}. \end{aligned}$$

The coordinates of these SPs are $y = 3, {}^3\!/_4, 1$. Hence, the range of f is $[{}^3\!/_4, 3]$. This gives the range of the original fraction as $[{}^1\!/_3, {}^4\!/_3]$.

2684. Taking out a factor of $(x^2-1)^4$,

$$\begin{aligned} & (x^2-1)^5 + (x^2-1)^4(2x-3) = 0 \\ \Longrightarrow\ & (x^2-1)^4\big((x^2-1)+(2x-3)\big) = 0 \\ \Longrightarrow\ & (x-1)^4(x+1)^4(x^2+2x-4) = 0. \end{aligned}$$

Therefore, $x = \pm 1$, or $x^2 + 2x - 4 = 0$. The quadratic formula gives $x = -1 \pm \sqrt{5}$.

2685. An invertible function is one-to-one, which means that no line $y = p$ crosses the curve $y = \text{f}(x)$ more than once.

The function f is not invertible on $\mathbb{R}$, so $y = \text{f}(x)$ must have at least one turning point. But, by the same token, $y = \text{f}(x)$ has no turning points on the invertible domains $(-\infty, k]$ and $[k, \infty)$. So, the turning point must be at $x = k$. A turning point is also a stationary point. QED.

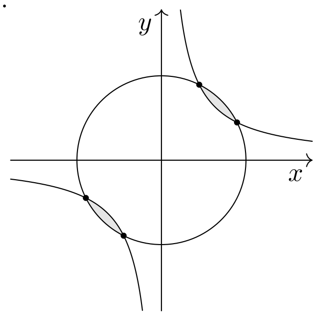
2686. Knowing $X = Y$ gives us no information about X itself, only that Y has come out the same. Hence, $\mathbb{P}(X + Z) = 7$ is unchanged by the information.

2687. (a) Substituting $y = \frac{2}{x}$,

$$\begin{aligned} & x^2 + \tfrac{4}{x^2} = 5 \\ \Longrightarrow\ & x^4 - 5x^2 + 4 = 0 \\ \Longrightarrow\ & x = \pm 1, \pm 2. \end{aligned}$$

Intersections at $(\pm 1, \pm 2)$ and $(\pm 2, \pm 1)$.

(b) The product needs to be negative, so we need exactly one factor to be negative. This only occurs when outside the hyperbola, but inside the circle:



2688. (a) Defining θ to be the acute angle between the F N force and the 200 N force, we resolve parallel and perpendicular to the 200 N force:

$$\begin{aligned} F\cos\theta - 200 &= 50 \times 2.6\cos\theta, \\ F\sin\theta - 120 &= 50 \times 2.6\sin\theta. \end{aligned}$$

Rearranging,

$$\begin{aligned} (F - 130)\cos\theta &= 200, \\ (F - 130)\sin\theta &= 120. \end{aligned}$$

Dividing these, $\tan\theta = {}^{120}\!/_{200} = {}^3\!/_5$. So, the obtuse angle is $\pi - \arctan {}^3\!/_5$, as required.

(b) Substituting back in, $F = 363$ (3sf).

2689. We set $u = x$ and $\frac{dv}{dx} = \sin x$. Then $\frac{du}{dx} = 1$ and $v = -\cos x$. Substituting into the parts formula,

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} x \sin x \, dx \\ &= \Big[-x\cos x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos x \, dx \\ &= \Big[-x \cos x + \sin x\Big]_0^{\frac{\pi}{2}} \\ &= (1) - (0) \\ &= 1, \text{ as required.} \end{aligned}$$

2690. This is false. In general, a set of three equations in two unknowns does not yield any solutions. But such a thing is possible in special cases. Consider the set of lines

$$\begin{aligned} y &= x, \\ y &= 2x, \\ y &= 3x. \end{aligned}$$

These are concurrent at the origin, i.e. they have a unique solution point (x, y).

2691. The individual terms have been exponentiated, rather than the entire LHS and RHS. This gives an error on the RHS. The corrected algebra is

$$\begin{aligned} &\ln(x^2 + y - 1) = \ln x + \ln y \\ \Longrightarrow\ & x^2 + y - 1 = e^{\ln x + \ln y} \\ \Longrightarrow\ & x^2 + y - 1 = xy. \end{aligned}$$

2692. (a) Consider a transformed graph with minima at $(3, 0)$ and $(5, 0)$. These are points of tangency with the x axis, so double roots. Hence, the (monic) graph is

$$y = (x - 3)^2 (x - 5)^2.$$

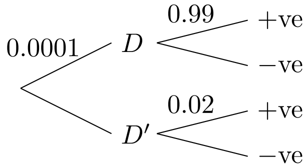
Translating this by $\mathbf{j}$, the equation is

$$y = (x - 3)^2 (x - 5)^2 + 1.$$

(b) The points $(3, 1)$ and $(5, 1)$ are symmetrical, so the local maximum must be halfway between them. This gives a maximum at $(4, 2)$.

2693. The graph $y = \mathrm{f}(x)$ is a parabola, with a vertex at $x = k$. Each side of and including this point, the function f is one-to-one and thus invertible. Hence, we can define $D_1 = (-\infty, k]$ and $D_2 = [k, \infty)$. The union is $\mathbb{R}$, as required.

2694. (a) Conditioning on having/not have the disease:

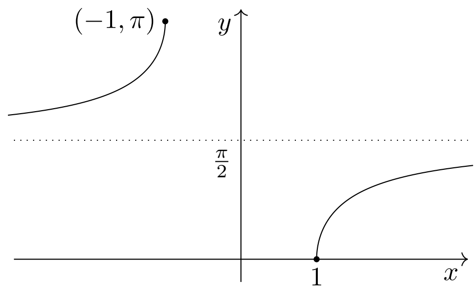


(b) Restricting the possibility space to the two +ve branches,

$$\begin{aligned} &\mathbb{P}(D \mid +) \\ &= \frac{0.0001 \times 0.99}{0.0001 \times 0.99 + 0.9999 \times 0.02} \\ &= \frac{1}{203}. \end{aligned}$$

(c) Of 203 people who test +ve, we would expect only one to have the disease. This is very low. This problem occurs when screening *randomly* (as opposed to when symptoms appear) for rare conditions: you are much more likely to get a false positive from the huge population without the disease than a true positive from the tiny population with the disease.

2695. The secant function is invertible on $[0, \pi] \setminus \{\pi/2\}$, with codomain $\mathbb{R} \setminus (-1, 1)$. Switching inputs for outputs, the arcsec function has domain $\mathbb{R} \setminus (-1, 1)$ and codomain $[0, \pi] \setminus \{\pi/2\}$:



2696. The reaction force between the stacked blocks is $R_1 = m_1 g$ and between the lower block and the table is $R_2 = (m_1 + m_2)g$. Hence, the maximum value F_{max} of friction at the table is greater than maximum value of friction between the blocks:

$$\mu(m_1 + m_2)g \geq \mu m_1 g.$$

The only force which can accelerate the lower block is the friction at its upper surface. As shown above, this will always be counteracted by friction on the lower surface. Hence, the lower stacked block will not accelerate. □

2697. If $s \in S$, then $\mathrm{f}(s) = 0$ and $\mathrm{g}(s) = 0$, so $\mathrm{f}(s) = \mathrm{g}(s)$. So, since any element $s \in S$ satisfies equation E, everything in S is in the solution set of E.

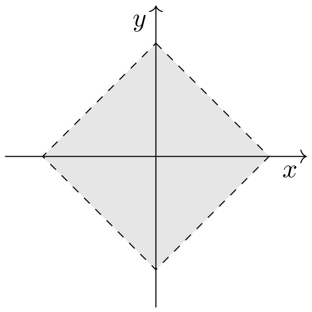
But the reverse isn't true: there can be roots of E which are not in S. For instance, $\mathrm{f}(x) = x$ and $\mathrm{g}(x) = x^2$ both have $S = \{0\}$, but $x^2 = x$ also has $x = 1$ as a root.

(a) False,

(b) True,

(c) False.

2698. The boundary equations are $x + y = \pm 1$ and $x - y = \pm 1$, which form a square:



2699. Starting with the LHS,

$$\begin{aligned} & \cos x \cos 2x + \sin x \sin 2x \\ \equiv\ & \cos x(1 - 2\sin^2 x) + \sin x(2 \sin x \cos x) \\ \equiv\ & \cos x - 2\cos x \sin^2 x - 2\sin^2 x \cos x \\ \equiv\ & \cos x, \text{ as required.} \end{aligned}$$

2700. Let $u = x + 1$. Then $du = dx$ and $x = u - 1$. Enacting the substitution,

$$\begin{aligned} & \int \frac{2x^2}{x+1}\, dx \\ = & \int \frac{2(u-1)^2}{u}\, du. \end{aligned}$$

Multiplying the numerator out and splitting the fraction up, this is

$$\begin{aligned} & \int 2u - 4 + 2u^{-1}\, du \\ \equiv\ & u^2 - 4u + 2\ln|u| + c. \end{aligned}$$

Re-substituting for u and subsequently choosing a new constant of integration, the integral is

$$\begin{aligned} & (x+1)^2 - 4(x+1) + 2\ln|x+1| + c \\ = & x^2 - 2x + 2\ln|x+1| + d. \end{aligned}$$

——— End of 27th Hundred ———